Solutions

3.1	The pressure is given by the hydrostatic pressure $p(x,z) = \rho_{\text{ice}} g (H(x) - z)$, which is zero at the surface.	0.3

3.2a	The outward force on a vertical slice at a distance x from the middle and of a given width Δy is obtained by integrating up the pressure times the area: $$F(x) = \Delta y \int_0^{H(x)} \rho_{\text{ice}}\, g\, (H(x) - z)\, \mathrm{d}z = \frac{1}{2}\, \Delta y\, \rho_{\text{ice}}\, g\, H(x)^2$$ which implies that $\Delta F = F(x) - F(x + \Delta x) = -\frac{\mathrm{d}F}{\mathrm{d}x} \Delta x = -\Delta y\, \rho_{\text{ice}}\, g\, H(x) \frac{\mathrm{d}H}{\mathrm{d}x} \Delta x$. This finally shows that $$S_{\text{b}} = \frac{\Delta F}{\Delta x \Delta y} = -\,\rho_{\text{ice}}\, g\, H(x) \frac{\mathrm{d}H}{\mathrm{d}x}$$ Notice the sign, which must be like this, since S_b was defined as positive and $H(x)$ is a decreasing function of x.	0.9
3.2b	To find the height profile, we solve the differential equation for $H(x)$: $$-\frac{S_{\text{b}}}{\rho_{\text{ice}}\, g} = H(x) \frac{\mathrm{d}H}{\mathrm{d}x} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} H(x)^2$$ with the boundary condition that $H(L) = 0$. This gives the solution: $$H(x) = \sqrt{\frac{2 S_b L}{\rho_{\text{ice}}\, g}} \sqrt{1 - x/L}$$ Which gives the maximum height $H_{\text{m}} = \sqrt{\frac{2 S_b L}{\rho_{\text{ice}}\, g}}$. Alternatively, dimensional analysis could be used in the following manner. First notice that $\mathcal{L} = [H_{\text{m}}] = [\rho_{\text{ice}}^{\alpha}\, g^{\beta} \tau_{\text{b}}^{\gamma} L^{\delta}]$. Using that $[\rho_{\rho_{\text{ice}}}] = \mathcal{M}\mathcal{L}^{-3}$, $[g] = \mathcal{L}\mathcal{T}^{-2}$, $[\tau_b] = \mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-2}$, demands that $\mathcal{L} = [H_{\text{m}}] = [\rho_i{}^{\alpha} g^{\beta} \tau_b{}^{\gamma} L^{\delta}] = \mathcal{M}^{\alpha+\gamma} \mathcal{L}^{-3\alpha+\beta-\gamma+\delta}\, \mathcal{T}^{-2\beta-2\gamma}$, which again implies $\alpha + \gamma = 0$, $-3\alpha + \beta - \gamma + \delta = 1$, $2\beta + 2\gamma = 0$. These three equations are solved to give $\alpha = \beta = -\gamma = \delta - 1$, which shows that $$H_{\text{m}} \propto \left(\frac{S_{\text{b}}}{\rho_{\rho_{\text{ice}}} g}\right)^{\gamma} L^{1-\gamma}$$ Since we were informed that $H_{\text{m}} \propto \sqrt{L}$, it follows that $\gamma = 1/2$. With the boundary condition $H(L) = 0$, the solution then take the form $$H(x) \propto \left(\frac{S_{\text{b}}}{\rho_{\text{ice}}\, g}\right)^{1/2} \sqrt{L - x}$$ The proportionality constant of $\sqrt{2}$ cannot be determined in this approach.	0.8

3.2c	For the rectangular Greenland model, the area is equal to $A = 10L^2$ and the volume is found by integrating up the height profile found in problem 3.2b: $V_{\mathrm{G,ice}} = (5L)2\int_0^L H(x)\,\mathrm{d}x = 10L\int_0^L \left(\frac{\tau_\mathrm{b} L}{\rho_\mathrm{ice}\, g}\right)^{1/2} \sqrt{1-x/L}\,\mathrm{d}x = 10H_\mathrm{m}L^2 \int_0^1 \sqrt{1-\tilde{x}}\,\mathrm{d}\tilde{x}$ $= 10H_\mathrm{m}L^2\left[-\frac{2}{3}(1-\tilde{x})^{3/2}\right]_0^1 = \frac{20}{3}H_\mathrm{m}L^2 \propto L^{5/2}$, where the last line follows from the fact that $H_\mathrm{m} \propto \sqrt{L}$. Note that the integral need not be carried out to find the scaling with L. This implies that $V_{\mathrm{G,ice}} \propto {A_G}^{5/4}$ and the wanted exponent is $\gamma = 5/4$.	0.5

3.3	According to the assumption of constant accumulation c the total mass accumulation rate from an area of width Δy between the ice divide at $x = 0$ and some point at $x > 0$ must equal the total mass flux through the corresponding vertical cross section at x. That is: $\rho c x \Delta y = \rho \Delta y H_\mathrm{m} v_x(x)$, from which the velocity is isolated: $$v_x(x) = \frac{cx}{H_\mathrm{m}}$$	0.6

3.4	From the given relation of incompressibility it follows that $$\frac{\mathrm{d}v_z}{\mathrm{d}z} = -\frac{\mathrm{d}v_x}{\mathrm{d}x} = -\frac{c}{H_\mathrm{m}}$$ Solving this differential equation with the initial condition $v_z(0) = 0$, shows that: $$v_z(z) = -\frac{cz}{H_\mathrm{m}}$$	0.6

3.5	Solving the two differential equations $$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{cz}{H_\mathrm{m}} \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{cx}{H_\mathrm{m}}$$ with the initial conditions that $z(0) = H_\mathrm{m}$, and $x(0) = x_i$ gives $$z(t) = H_\mathrm{m}\,\mathrm{e}^{-ct/H_\mathrm{m}} \quad \text{and} \quad x(t) = x_i\,\mathrm{e}^{ct/H_\mathrm{m}}$$ This shows that $z = H_\mathrm{m}\,x_i\,/x$, meaning that flow lines are hyperbolas in the xz-plane. Rather than solving the differential equations, one can also use them to show that $$\frac{\mathrm{d}}{\mathrm{d}t}(xz) = \frac{\mathrm{d}x}{\mathrm{d}t}z + x\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{cx}{H_\mathrm{m}}z - x\frac{cz}{H_\mathrm{m}} = 0$$ which again implies that $xz = \text{const}$. Fixing the constant by the initial conditions, again leads to the result that $z = H_\mathrm{m}x_i/x$.	0.9

3.6	At the ice divide, $x = 0$, the flow will be completely vertical, and the t-dependence of z found in 3.5 can be inverted to find $\tau(z)$. One finds that $\tau(z) = \frac{H_\mathrm{m}}{c}\ln\left(\frac{H_\mathrm{m}}{z}\right)$.	1.0

3.7a	The present interglacial period extends to a depth of 1492 m, corresponding to 11,700 year. Using the formula for $\tau(z)$from problem 3.6, one finds the following accumulation rate for the interglacial: $$c_{\text{ig}} = \frac{H_\text{m}}{11{,}700\text{ years}} \ln\left(\frac{H_\text{m}}{H_\text{m} - 1492\text{ m}}\right) = 0.1749\text{ m/year}.$$ The beginning of the ice age 120,000 years ago is identified as the drop in $\delta^{18}\text{O}$ in figure 3.2b at a depth of 3040 m. Using the vertical flow velocity found in problem 3.4, on has $\frac{\text{d}z}{z} = -\frac{c}{H_\text{m}}\text{d}t$, which can be integrated down to a depth of 3040 m, using a stepwise constant accumulation rate: $$H_\text{m} \ln\left(\frac{H_\text{m}}{H_\text{m} - 3040\text{ m}}\right) = -H_\text{m}\int_{H_\text{m}}^{H_\text{m}-3040\text{ m}} \frac{1}{z}\text{d}z$$ $$= \int_{11{,}700\text{ year}}^{120{,}000\text{ year}} c_{\text{ia}}\,\text{d}t + \int_0^{11{,}700\text{ year}} c_{\text{ig}}\,\text{d}t$$ $$= c_{\text{ia}}(120{,}000\text{ year-}11{,}700\text{ year}) + c_{\text{ig}}11{,}700\text{ year}$$ Isolating form this equation leads to $c_{\text{ia}} = 0.1232$, i.e. far less precipitation than now.	0.8
3.7b	Reading off from figure 3.2b: $\delta^{18}\text{O}$ changes from $-43{,}5$ ‰ to $-34{,}5$ ‰. Reading off from figure 3.2a, T then changes from -40 °C to -28 °C. This gives $\Delta T \approx 12$ °C.	0.2

3.8	From the area A_G one finds that $L = \sqrt{A_\text{G}/10} = 4.14 \times 10^5$ m. Inserting numbers in the volume formula found in 3.2c, one finds that: $$V_{\text{G,ice}} = \frac{20}{3} L^{5/2} \sqrt{\frac{2S_\text{b}}{\rho_{\text{ice}}g}} = 3.45 \times 10^{15}\text{ m}^3$$ This ice volume must be converted to liquid water volume, by equating the total masses, i.e. $V_{\text{G,wa}} = V_{\text{G,ice}}\frac{\rho_{\text{ice}}}{\rho_{\text{wa}}} = 3.17 \times 10^{15}\text{ m}^3$, which is finally converted to a sea level rise, as $h_{\text{G,rise}} = \frac{V_{\text{G,wa}}}{A_\text{o}} = 8.79$ m.	0.6

3.9 (1.6)

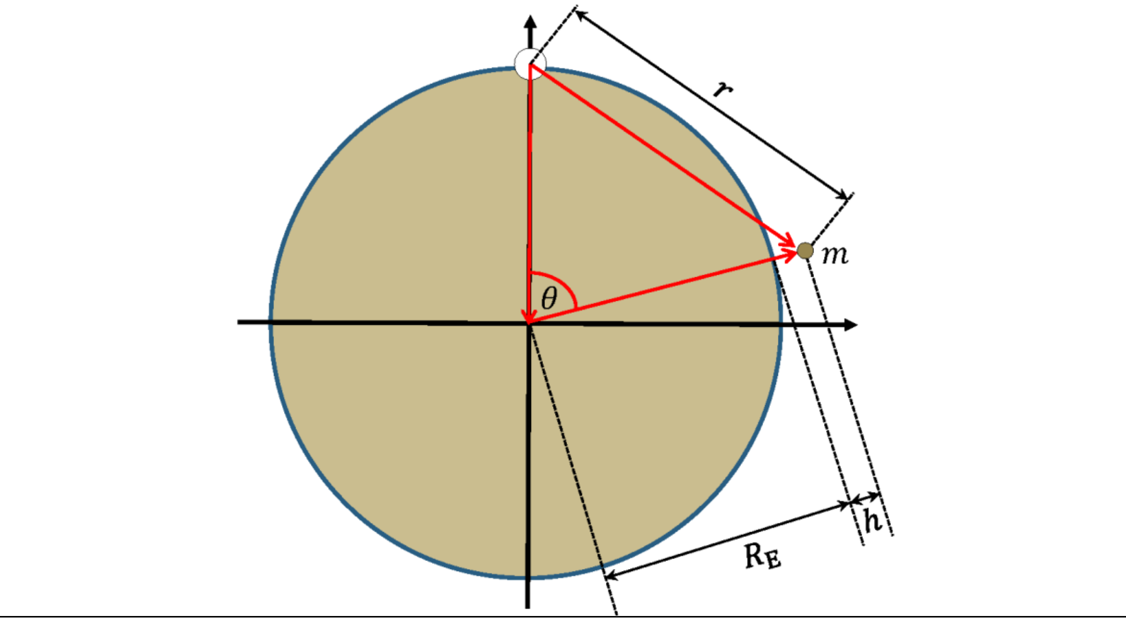

Figure 3.S1 Geometry of the ice ball (white circle) with a test mass m (small gray circle).

The total mass of the ice is

$$M_{\text{ice}} = V_{\text{G,ice}}\,\rho_{\text{ice}} = 3.17 \times 10^{18}\ \text{kg} = 5.31 \times 10^{-7} m_{\text{E}}$$

The total gravitational potential felt by a test mass m at a certain height h above the surface of the Earth, and at a polar angle θ (cf. figure 3.S1), with respect to a rotated polar axis going straight through the ice sphere is found by adding that from the Earth with that from the ice:

$$U_{\text{tot}} = -\frac{Gm_{\text{E}}m}{R_{\text{E}}+h} - \frac{GM_{\text{ice}}m}{r} = -mgR_E\left(\frac{1}{1+h/R_E} + \frac{M_{ice}/m_E}{r/R_E}\right)$$

where $g = Gm_E/R_E^2$. Since $h/R_{\text{E}} \ll 1$ one may use the approximation given in the problem, $(1+\text{x})^{-1} \approx 1-x,\ \ |x| \ll 1$, to approximate this by

$$U_{\text{tot}} \approx -mgR_E\left(1 - \frac{h}{R_E} + \frac{M_{ice}/m_E}{r/R_E}\right).$$

Isolating h now shows that $h = h_0 + \frac{M_{ice}/m_E}{r/R_E}R_E$, where $h_0 = R_E + U_{\text{tot}}/(mg)$. Using again that $h/R_{\text{E}} \ll 1$, trigonometry shows that $r \approx 2R_{\text{E}}|\sin(\theta/2)|$, and one has:

$$h(\theta) - h_0 \approx \frac{M_{\text{ice}}/m_{\text{E}}}{2|\sin(\theta/2)|}R_E \approx \frac{1.69\ \text{m}}{|\sin(\theta/2)|}.$$

To find the magnitude of the effect in Copenhagen, the distance of 3500 km along the surface is used to find the angle $\theta_{\text{CPH}} = (3.5 \times 10^6\ \text{m})/R_E \approx 0.549$, corresponding to $h_{\text{CPH}} - h_0 \approx 6.25$ m. Directly opposite to Greenland corresponds to $\theta = \pi$, which gives $h_{\text{OPP}} - h_0 \approx 1.69$ m. The difference is then $h_{\text{CPH}} - h_{\text{OPP}} \approx 4.56$ m, where h_0 has dropped out.

	Total	**9.0**