Problem 2. (Solution):

2.1: Suppose that the density of neutron stars is the same than the density of heavy nuclei: $\rho = \frac{m}{V} = \frac{A \cdot u}{\frac{4}{3} r^3 \pi} = \frac{A \cdot u}{\frac{4}{3} r_0^3 A \pi} = \frac{3u}{4\pi r_0^3} = 2{,}3 \cdot 10^{17} \frac{\text{kg}}{\text{m}^3}$, where u is the atomic mass unit.

2.2: Considering the equilibrium of a particle with mass Δm at the equator of the neutron star at the maximal angular velocity the gravitational force balances with the centrifugal force acting on it:

$$G \frac{\Delta m M}{r^2} = \Delta m r \omega_{\max}^2$$

$$\omega_{\max} = \sqrt{\frac{GM}{r^3}} = \sqrt{\frac{G\rho \frac{4}{3} r^3 \pi}{r^3}} = \sqrt{\frac{4\pi G \rho}{3}} = 8000 \, \frac{\text{rad}}{\text{s}}.$$

2.3: Let us collect the SI units of the physical quantities in the formula:

$[P] = \text{VA}$; $[\mu_0] = \frac{\text{Vs}}{\text{Am}}$; $[\ddot{\boldsymbol{m}}] = \frac{\text{Am}^2}{\text{s}^2}$; $[\omega] = \frac{1}{\text{s}}$; $[c] = \frac{\text{m}}{\text{s}}$.

Using the dimensional analysis we get

$\text{V}^{\alpha} \, \text{A}^{-\alpha+\beta} \, \text{m}^{-\alpha+2\beta+\delta} \, \text{s}^{\alpha-2\beta-\gamma-\delta} = \text{VA}.$

It means that $\alpha = 1$; $\beta = 2$; $\gamma = 0$; $\delta = -3$. So $P = \frac{1}{6\pi} \mu_0 |\ddot{\boldsymbol{m}}|^2 c^{-3}$.

2.4: The power of radiation of the pulsar is equal to the decrease of rotational kinetic energy of the neutron star:

$$P = -\frac{1}{6\pi} \mu_0 (\ddot{\boldsymbol{m}})^2 c^{-3} = \frac{\mathrm{d}\left(\frac{1}{2} I \omega^2\right)}{\mathrm{d}t},$$

where $I = \frac{2}{5} M R^2$ is the moment of inertia of the neutron star (pulsar). Taking into consideration that $\dot{\omega} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\pi}{T}\right) = -2\pi \frac{1}{T^2} \dot{T}$, we get

$$P = \frac{\mathrm{d}\left(\frac{1}{2} I \omega^2\right)}{\mathrm{d}t} = \frac{2}{5} M R^2 \omega \dot{\omega} = -\frac{2}{5} M R^2 \frac{(2\pi)^2}{T^3} \dot{T} = -\frac{1}{6\pi} \mu_0 (\ddot{\boldsymbol{m}})^2 c^{-3}.$$

The rotational axis of the star is perpendicular to the momentary direction of its magnetic momentum $\boldsymbol{m}$, and it should be exploited accordingly: $\dot{\boldsymbol{m}} = \boldsymbol{\omega} \times \boldsymbol{m}$ and $\ddot{\boldsymbol{m}} = -\omega^2 \boldsymbol{m}$.

Finally, we can express $\dot{T}$:

$$\dot{T} = \frac{5\pi}{3} \mu_0 \frac{(\boldsymbol{m})^2}{M R^2 T c^3}.$$

2.5: The density of Crab pulsar is $\rho = \frac{M}{V} = \frac{1.4 \cdot 1.989 \cdot 10^{30}\ \mathrm{kg}}{\frac{4}{3}\pi(10^4\ \mathrm{m})^3} = 6.65\ \cdot 10^{17}\ \frac{\mathrm{kg}}{\mathrm{m}^3}$, which is really much larger than the heavy nucleus density. The maximal possible angular velocity of Crab pulsar was

$$\omega_{\max} = \sqrt{\frac{4\pi G\rho}{3}} = 13\,600\ \frac{\mathrm{rad}}{\mathrm{s}},\ \text{so } T_{\min} = \frac{2\pi}{\omega_{\max}} = 4.62\ \cdot\ 10^{-4}\ \mathrm{s}.$$

At the moment the period of Crab pulsar is $T = 330 \cdot 10^{-4}$ s, which is much larger than its possible minimal value, and now $\dot{T} = 4\ \cdot\ 10^{-13}$ (which is dimensionless).
Let us use the result of 2.4:

$$T\dot{T} = \frac{5\pi}{3}\mu_0 \frac{(\boldsymbol{m})^2}{MR^2c^3} = K = (330 \cdot 10^{-4}\ \mathrm{s}) \cdot (4\ \cdot\ 10^{-13}) = 1.32\ \cdot\ 10^{-14}\ \mathrm{s},$$

where K is the notation of the constant in the equation. This equation is a separable differential equation, so

$$\int_{4.62 \cdot 10^{-4}\ \mathrm{s}}^{3.3 \cdot 10^{-2}\ \mathrm{s}} T\ \mathrm{d}T = K\ t,$$

where t is the possible longest age of crab pulsar. After the integration we get

$$\left.\frac{T^2}{2}\right]_{4.62 \cdot 10^{-4}\ \mathrm{s}}^{3.3 \cdot 10^{-2}\ \mathrm{s}} = K\ t,$$

which yields $t = 4.1 \cdot 10^{10}$ s = 1300 years. (The lower limit of integration is negligible.)

The parametric solution is $t = \frac{T^2}{2K} = \frac{3MR^2c^3}{10\pi\mu_0(\boldsymbol{m})^2}T^2$.

Note: This result is not a false guess because Crab Nebula is a supernova remnant which was observed by Chinese astronomers in 1054 (so almost 1000 years ago). So Crab pulsar is 960 years old now.

2.6: The magnetic induction vector $\boldsymbol{B}$ along the line of action of the magnetic momentum is

$$\boldsymbol{B} = \frac{\mu_0}{2\pi}\frac{\boldsymbol{m}}{r^3}.$$

You can proof this formula using the Biot-Savart law or using analogy between electric dipoles and magnetic dipoles.

We can get the value of $\boldsymbol{m}$ from the expression of constant K:

$$|\boldsymbol{m}| = \sqrt{\frac{3T\dot{T}MR^2c^3}{5\pi\mu_0}} = 8.28\ \cdot\ 10^{29}\ \mathrm{Am}^2.$$

So we can find the maximal value of the magnetic field on the surface of the pulsar:

$$\boldsymbol{B} = \frac{\mu_0}{2\pi}\frac{\boldsymbol{m}}{R^3} = 1.66 \cdot 10^{11}\ \mathrm{T}.$$

2.7: Let us follow the same calculation as in the semi-classical Bohr's model. We suppose that the wavelength of the electron at the lowest level (ground state) is the circumference of the circular orbit:

$$\lambda = 2r\pi = \frac{h}{mv}.$$

The Lorentz force gives the centripetal force:

$$m\frac{v^2}{r} = eBv,$$

which yields the radius of the orbit of the electron: $r = \frac{mv}{eB}$. Let us insert this radius into the previous formula of de Broglie wavelength:

$$2\pi\frac{mv}{eB} = \frac{h}{mv}.$$

We can find the critical magnetic field B_{crit} if we substitute c (speed of light) into the formula above instead of v (speed of the electron):

$$B_{\mathrm{crit}} = \frac{2\pi m^2 c^2}{he} = \frac{m^2 c^2}{\hbar e} = 4.4 \cdot 10^9\ \mathrm{T}.$$

Note: This is the so-called Schwinger limit. In quantum electrodynamics (QED), the Schwinger limit is a scale above which the electromagnetic field is expected to become nonlinear.

You can see that the magnetic field on the surface of the pulsar is much larger (almost 40 times larger) than the Schwinger limit. This explains the unusual spectra of the pulsars.