Problem 3. (Solution):

3.1: Near the point A, outside the small sphere the electric field is the superposition of E and the field of a small electric dipole:

$$E_{\text{out}} = E - \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}}{R^3} = E - \frac{1}{4\pi\varepsilon_0}\frac{\vec{P}V}{R^3} = E - \frac{1}{4\pi\varepsilon_0}\frac{\varepsilon_0(\varepsilon_{\text{r}}-1)E_{\text{in}}\frac{4\pi}{3}R^3}{R^3} =$$

$$= E - \frac{(\varepsilon_{\text{r}}-1)E_{\text{in}}}{3}$$

This field should be equal to the homogenous field E_{in} inside the sphere:

$$E_{\text{out}} = E_{\text{in}} = E - \frac{(\varepsilon_{\text{r}}-1)E_{\text{in}}}{3} \quad \rightarrow \quad E_{\text{in}} = \frac{3}{\varepsilon_{\text{r}}+2}E.$$

Let us insert this into the expression of the dipole momentum:

$$p = P\cdot V = [\varepsilon_0(\varepsilon_{\text{r}}-1)E_{\text{in}}]\cdot\left(\frac{4\pi}{3}R^3\right) = \varepsilon_0(\varepsilon_{\text{r}}-1)\cdot\frac{3}{\varepsilon_{\text{r}}+2}E\cdot\frac{4\pi}{3}R^3$$
$$= \frac{4\pi\varepsilon_0(\varepsilon_{\text{r}}-1)R^3}{\varepsilon_{\text{r}}+2}E.$$

It yields

$$\alpha = \frac{4\pi\varepsilon_0(\varepsilon_{\text{r}}-1)R^3}{\varepsilon_{\text{r}}+2}.$$

3.2: The relationship between force and potential energy: $\vec{F} = -\text{grad}\,U$. It means that

$$\vec{F}_x = -\frac{\delta}{\delta x}(U) = -\frac{\delta}{\delta x}\left(-\frac{1}{2}\alpha\boldsymbol{E}^2\right) = \frac{\alpha}{2}\frac{\delta}{\delta x}\left(E_x^2+E_y^2+E_y^2\right)$$
$$= \alpha E_x\frac{\partial E_x}{\partial x} + \alpha E_y\frac{\partial E_y}{\partial x} + \alpha E_z\frac{\partial E_z}{\partial x} = p_x\frac{\partial E_x}{\partial x} + p_y\frac{\partial E_y}{\partial x} + p_z\frac{\partial E_z}{\partial x}.$$

3.3: The intensity can be described with the Poynting vector:

$$\bar{I} = I(x,y,z) = \overline{|\vec{S}|} = \overline{\left|\vec{E}\times\frac{1}{\mu_0}\vec{B}\right|} = \frac{1}{\mu_0 c}\overline{|\vec{E}|^2} = \varepsilon_0 c\overline{|\vec{E}|^2} = \frac{1}{2}\varepsilon_0 c[E(x,y,z)]^2.$$

3.4: Let us use equation (2):

$$\vec{F} = \langle\vec{F}(t)\rangle = \langle(\vec{p}(t)\cdot\text{grad})\vec{E}(t)\rangle = \langle\left(\alpha\vec{E}(t)\cdot\text{grad}\right)\vec{E}(t)\rangle =$$
$$= \frac{\alpha}{2}\langle\text{grad}\,\vec{E}^2(\text{t})\rangle = \frac{\alpha}{2}\text{grad}\langle\vec{E}^2(\text{t})\rangle = \frac{\alpha}{2}\text{grad}\frac{E_{\max}^2}{2} = \frac{\alpha}{2\varepsilon_0 c}\text{grad}\,I.$$

It means that

$$\gamma = \frac{\alpha}{2\varepsilon_0 c}.$$

3.5:

$$F_y = \gamma \frac{\mathrm{d}I}{\mathrm{d}y} = -\frac{\alpha I_0}{\varepsilon_0 c b^2}\, y = m\ddot{y}$$

It means that it is a simple harmonic motion with the amplitude *d* and with angular frequency

$$\omega = \sqrt{\frac{\alpha I_0}{\varepsilon_0 c b^2 m}}\,.$$

3.6:

$$F^{\mathrm{rad}} = \frac{P^{\mathrm{rad}}}{c} = \frac{\mu_0 \omega^4}{12\pi c^2}\alpha^2 E^2 = \frac{\mu_0 \omega^4 \alpha^2}{12\pi c^2}\,\frac{2I}{\varepsilon_0 c} = \frac{\omega^4 \alpha^2 I}{6\pi \varepsilon_0^2 c^5}\,.$$

3.7: The condition of equilibrium is

$$F^{\mathrm{rad}} + \gamma \frac{\mathrm{d}I}{\mathrm{d}x} = 0\,,$$

where

$$F^{\mathrm{rad}} = \frac{\omega^4 \alpha^2}{6\pi \varepsilon_0^2 c^5} I_0 \left(1 - \frac{\xi^2}{a^2}\right) \quad \text{and} \quad \gamma \frac{\mathrm{d}I}{\mathrm{d}x} = -\frac{\alpha}{\varepsilon_0 c} I_0 \frac{\xi}{a^2}\,.$$

This is a quadratic equation for ξ:

$$\frac{\omega^4 \alpha}{6\pi \varepsilon_0 c^4}\left(a^2 - \xi^2\right) - \xi = 0\,.$$

The fraction in the equation has a dimension of 1/m, so let us denote it as $1/x_0$:

$$x_0 = \frac{6\pi\varepsilon_0 c^4}{\omega^4 \alpha} = \frac{6\pi\varepsilon_0 c^4}{\omega^4}\,\frac{\varepsilon_{\mathrm{r}} + 2}{4\pi\varepsilon_0(\varepsilon_{\mathrm{r}} - 1)R^3} = \frac{3(\varepsilon_{\mathrm{r}} + 2)c^4}{2(\varepsilon_{\mathrm{r}} - 1)\omega^4 R^3}\,,$$

where $\frac{c}{\omega} = \frac{\lambda}{2\pi}$, so

$$x_0 = \frac{3(\varepsilon_{\mathrm{r}} + 2)\lambda^4}{2(2\pi)^4(\varepsilon_{\mathrm{r}} - 1)R^3} = 2.885\ \mathrm{mm}\,.$$

We can write the quadratic equation in this way:

$$\xi^2 + x_0 \xi - a^2 = 0\,,$$

and its positive root is

$$\xi = \frac{\sqrt{x_0^2 + 4a^2} - x_0}{2} \approx \frac{a^2}{x_0} = 139\ \mathrm{nm}.$$